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A Path Integral Formula for Certain Fourth-Order Elliptic Operators

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Abstract. We define a discretized path integral formula for the operator $-\Delta^2 - V$. This formula is the generalization of the Feynman–Kac formula for $+\Delta - V$.

1. Introduction

In this Letter, we want to construct the analogue of Brownian motion and of the Feynman–Kac formula for a fourth-order elliptic operator like $-\Delta^2 - V$, where Δ is the usual Laplace operator in $\mathbb{R}^d$ and V is a potential. There have been several attempts at constructing such objects in the past. Let us cite, for example, the general construction of Ehrenpreis [1] using a Fourier transform approach, and the construction of Daletskii [2]. More recently, a very detailed study of this kind of path integral was made by Hochberg [3]. The approach of Hochberg is also a Fourier transform approach and he constructs a measure (which is of infinite total variation) on a space of real-valued paths. Last year, the second author constructed a positive probability measure on complex-valued paths valid for the operators $\mathrm{d}^n/\mathrm{d}x^n$ (see [4]).

Our approach here is different. We start from the Trotter formula for $-\Delta^2 - V$, and we want to 'disentangle' the exponential of $-\Delta^2$ and the exponential of $-V$ in the same spirit as in [5] and [6]. This can be done by rewriting the exponential of $-\Delta^2$ as an expectation value of an exponential of a first-order operator with stochastic coefficients, or what is the same as an expectation value of a translation operator, by a complex random variable.

Then, it is easy to disentangle the time-ordered product in Trotter's formula and obtain a Feynman–Kac formula. We are thus 'forced', by our formalism, to construct a complex-valued stochastic process on a Gaussian probability space, so that our construction is different from Hochberg's [3] or from any Fourier transform approach. In one variable, it gives back the construction of the second author [4] but starting from different ideas.

2. The Initial-Value Problem for Fourth-Order Elliptic Operators

We want to solve the initial-value problem for the evolution equation

$$\frac{\partial u}{\partial t} = -\tfrac{1}{8}\Delta^2 u - V(x)u, \quad u|_{t=0} = f(x), \tag{1}$$

where Δ is the usual Laplace operator in $\mathbb{R}^d$ and $V(x)$ is a real-valued function which is everywhere greater than 0. Moreover, the initial data $f(x)$ is a bounded continuous and integrable function to avoid any technical complication. It is well known that, under these conditions, the operator $-\frac{1}{8}\Delta^2 - V(x)$ is a negative self-adjoint operator and that the heat evolution semigroup is given by Trotter's formula, namely

$$\exp[-t(\tfrac{1}{8}\Delta^2 + V(x))] = \lim_{n\to\infty}\left(\exp\left[-\frac{t}{n}\frac{\Delta^2}{8}\right]\exp\left[-\frac{t}{n}V\right]\right)^n \tag{2}$$

At that level, this formula is almost obvious. The main reason for its validity is that the bracket

$$\left[\frac{1}{n}V, \frac{1}{n}\Delta^2\right]$$

is of order $O(1/n^2)$. We want now to disentangle this formula, namely to put on the left-hand side all the operators $e^{-(t/n)V}$ and on the right-hand all the operators $e^{-(t/n)(\Delta^2/8)}$. In the usual case for $\Delta/2$ instead of $-(\Delta^2/8)$, this is done by the usual Feynman–Kac formula, using a Brownian path, $x(t)$, with the heat kernel given by

$$\left\langle \exp\left[-\int_0^t V(x(s))\,\mathrm{d}s\right]\delta(x(t) - x)\,|\,x(0) = x_0\right\rangle, \tag{3}$$

where the expectation value $\langle\ \rangle$ is the conditional expectation of knowing that $x(0) = x_0$ over the Brownian paths $x(s)$ on the time interval $[0, t]$. We want to rewrite the evolution semi-group (2) in terms of a generalized path integral, as in (3).

3. The Case when $V \equiv 0$

As in [5] and [6], we view the path integral as a method for rewriting the exponential of the differential operator as an exponential of a first-order operator with the coefficient being an increment of the path. We introduce a sequence of independent Gaussian variables Γ_k and write the identity

$$\exp\left(-\frac{t}{n}\frac{\Delta^2}{8}\right) = \left\langle \exp\left(\frac{i}{2}\sqrt{\frac{t}{n}}\,\Delta\Gamma_k\right)\right\rangle, \tag{4}$$

where $\langle\ \rangle$ is the Gaussian average with respect to Γ_k.

Again, we introduce for each k, d independent Gaussian variables $G_k^{(1)}, \dots, G_k^{(d)}$

which are also independent of Γ_k. Then

$$\exp\left(\frac{i}{2}\sqrt{\frac{t}{n}}\,\Delta\Gamma_k\right) = \left\langle \exp\left[\left(\frac{1}{n}\right)^{1/4} |\Gamma_k|^{1/2} \exp\left[i\left(\frac{\pi}{4} + \frac{\operatorname{Arg}\Gamma_k}{2}\right)\right] \sum_{j=1}^{d} G_k^{(j)} \frac{\partial}{\partial x_j}\right]\right\rangle. \tag{5}$$

We then have, for any n, the identity

$$\exp\left(-\frac{t}{8}\Delta^2\right) = \left\langle \exp\left[\sum_{j=1}^{d}\left(\sum_{k=1}^{n}\left(\frac{t}{n}\right)^{1/4} |\Gamma_k|^{1/2} \exp\left[i\left(\frac{\pi}{4} + \frac{\operatorname{Arg}\Gamma_k}{2}\right)\right] G_k^{(j)}\right) \frac{\partial}{\partial x_j}\right]\right\rangle \tag{6}$$

where the sign $\langle\ \rangle$ is now an expectation over all Gaussian variables. This formula shows us that the value of the semigroup $\exp(-(t/8)\Delta^2)$ on a function f, is formally given by

$$\left(\exp\left(-\frac{1}{8}\Delta^2\right)f\right)(\mathbf{x}) = \left\langle f\left(\mathbf{x} + \sum_{k=1}^{n}\left(\frac{t}{n}\right)^{1/4} |\Gamma_k|^{1/2} \exp\left[i\left(\frac{\pi}{4} + \frac{\operatorname{Arg}\Gamma_k}{2}\right)\right] \mathbf{G}_k\right)\right\rangle \tag{7}$$

where

$$\mathbf{x} = (x_1, \ldots, x_d), \qquad \mathbf{G}_k = (G_k^{(1)}, \ldots, G_k^{(d)}).$$

Formula (7) has a meaning provided that f is a continuous bounded and integrable function on $\mathbb{R}^d$, which extends holomorphically in $\mathbb{C}^d$. It is to be noticed that the convergence of the expectation of the right-hand side of (7) is by cancellation, but it does converge as an oscillating integral. The stochastic process (with complex value)

$$\mathbf{X}^{(n)}(t) \equiv \mathbf{x} + \sum_{k=1}^{n}\left(\frac{t}{n}\right)^{1/4} |\Gamma_k|^{1/2} \exp\left[i\left(\frac{\pi}{4} + \frac{\operatorname{Arg}\Gamma_k}{2}\right)\right] \mathbf{G}_k \tag{8}$$

is the generalization of the discretized version of the Brownian path which would be

$$\mathbf{x} + \sum_{k=1}^{n}\left(\frac{t}{n}\right)^{1/2} \mathbf{G}_k .$$

It is clear that the Fourier Laplace transform of each infinitesimal increment of (7) is

given by

$$\left\langle \exp\left(\sum_{j=1}^{d} \xi_j \left(\frac{t}{n}\right)^{1/4} |\Gamma_k|^{1/2} \exp\left[i\left(\frac{\pi}{4} + \frac{\operatorname{Arg}\Gamma_k}{2}\right)\right] G_k^{(j)}\right)\right\rangle$$

$$= \int_{-\infty}^{+\infty} \int \exp\left[-\frac{x^2 + |y|^2}{2}\right] \exp\left(\frac{t}{n}\right)^{1/4} \exp\left[i\,\frac{\pi}{4}\right] \sqrt{x} \sum_{j=1}^{d} \xi_j y_j \,\frac{\mathrm{d}x\,\mathrm{d}y_1 \ldots \mathrm{d}y_d}{(2\pi)^{(d+1)/2}}$$

$$= \exp\left(-\frac{1}{8}\,\frac{t}{n}\,|\xi|^4\right). \tag{9}$$

This formula (9) is exactly the same as the one we should obtain if we were solving the initial-value problem (1) (for $V = 0$) by a Fourier transform with respect to space variables. In that case, the problem (1) becomes

$$\frac{\partial \hat{u}}{\partial t} = -\frac{1}{8}\left(\sum_{j=1}^{d} \xi_j^2\right)^2 \hat{u}$$

and the fundamental solution is just the last line of formula (9).

4. The Case of a General V

We are now ready to apply the same reasoning as in [5] to the solution of (1) for a general V. We start with the Trotter formula (2) in which we assume that V is a positive and continuous function on $\mathbb{R}^d$ which has a holomorphic extension to $\mathbb{C}^d$. We call the increment of (8)

$$\delta \mathbf{X}_k^{(n)} = \left(\frac{t}{n}\right)^{1/4} |\Gamma_k|^{1/2} \exp\left[i\left(\frac{\pi}{4} + \frac{\operatorname{Arg}\Gamma_k}{2}\right)\right] \mathbf{G}_k .$$

The Trotter formula (2) can be rewritten

$$\exp(-t(\tfrac{1}{8}\Delta^2 + V))$$

$$= \lim_{n\to\infty} \left\langle \exp(\delta\mathbf{X}_1^{(n)}\cdot\nabla)\exp\left(-\frac{t}{n}\,V\right)\exp(\delta\mathbf{X}_2^{(n)}\cdot\nabla)\exp\left(-\frac{t}{n}\,V\right)\ldots\right.$$

$$\left.\ldots\exp(\delta\mathbf{X}_n^{(n)}\cdot\nabla)\exp\left(-\frac{t}{n}\,V\right)\right\rangle,$$

but it is very easy to commute $\exp(\delta\mathbf{X}_k^{(n)}\cdot\nabla)$ with $\exp(-(t/n)V)$ because

$$\exp(\delta \mathbf{X}_k^{(n)} \cdot \nabla) \exp\left(-\frac{t}{n} V \right) = \exp\left(-\frac{t}{n} V(\cdot + \delta \mathbf{X}_k^{(n)}) \right) \exp(\delta \mathbf{X}_k^{(n)} \cdot \nabla)$$

and, thus, we obtain

$$\exp(-t(\tfrac{1}{8}\Delta^2 + V)) = \lim_{n\to\infty} \left\langle \exp\left(-\frac{t}{n} \sum_{k=1}^{n} V\left(\cdot + \sum_{l \leqslant k} \delta \mathbf{X}_l^{(n)} \right) \right) \times \right.$$
$$\left. \times \exp\left(\left(\sum_{l=1}^{n} \delta \mathbf{X}^{(n)} \right) \cdot \nabla \right) \right\rangle . \qquad (10)$$

This formula is a discretized version of a Feynman–Kac-type formula. The quantities

$$\mathbf{X}^{(n)}\left(\frac{kt}{n} \right) = \sum_{l \leqslant k} \delta \mathbf{X}_l^{(n)}$$

are exactly the discretized path taken from 0 to an intermediate time kt/n and, thus, on a bounded continuous and integrable function f on $\mathbb{R}^d$ which extends holomorphically to $\mathbb{C}^d$, (10) can be read as

$$(\exp(-t(\tfrac{1}{8}\Delta^2 + V))f)(\mathbf{x})$$
$$= \lim_{n\to\infty} \left\langle \exp\left(-\frac{t}{n} \sum_{k=1}^{n} V\left(\mathbf{x} + \mathbf{X}^{(n)}\left(\frac{kt}{n} \right) \right) \right) f(\mathbf{x} + \mathbf{X}^{(n)}(t)) \right\rangle . \qquad (11)$$

This formula (11) is a convergent integral, just because it is a rewriting of Trotter's formula (with a disentangling of the operators inside Trotter's formula).

5. Going to Continuum Limit

We now give a slight indication of how to go to the continuum limit for the paths introduced above. It is clear from the fact that $\exp(-(t/8)\Delta^2)$ is a semigroup, and from formula (7), that we can define a path in continuous time $\mathbf{x}(t)$ which is the weak limit of the discretized path (8). By 'weak limit', we understand here that the joint laws of the discretized paths at a sequence of time $t_1, \ldots, t_p$ will converge to a probability law defined on p-tuples if functions $f_1, \ldots, f_p$ which are bounded continuous and integrable on $\mathbb{R}^d$ and extend holomorphically on $\mathbb{C}^d$. A detailed study of the properties of the complex path will appear in [4]. We just note the following strange properties of the path $\mathbf{X}(t)$.

(1) Its increment are independent

(2) We have

$$\langle (\mathbf{X}(t))^p \rangle = \lim_{n\to\infty} \langle (\mathbf{X}^{(n)}(t))^p \rangle = 0 \quad \text{for } 1 \leqslant p \leqslant 3,$$

$$\langle (\mathbf{X}(t))^4 \rangle = \lim_{n\to\infty} \langle (\mathbf{X}^{(n)}(t))^4 \rangle = C_d t,$$

where C_d is a constant which depends only on d.

(3) $$\langle |\mathbf{X}(t)|^2 \rangle = \lim_{n \to +\infty} \langle |\mathbf{X}^{(n)}(t)|^2 \rangle = +\infty.$$

6. Conclusion

We have constructed a probability measure on a space of complex-valued paths which solves the evolution semigroup for an operator like $-\frac{1}{8}\Delta^2 - V$. The difference between our construction and Hochberg's [3] is that Hochberg constructs a signed measure (of infinite total variation) on a space of real-valued paths, so that in both cases we have to pay a price for going to fourth-order operators.

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